Generalized WDVV equations for F_4 pure N=2 Super-Yang-Mills theory

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Abstract

An associative algebra of holomorphic differential forms is constructed associated with pure N=2 Super-Yang-Mills theory for the Lie algebra F_4. Existence and associativity of this algebra, combined with the general arguments in the work of Marshakov, Mironov and Morozov, proves that the prepotential of this theory satisfies the generalized WDVV system.

1 Introduction

In 1994, Seiberg and Witten [1] solved the low energy behaviour of pure N=2 Super-Yang-Mills theory by giving the solution of the prepotential $\mathcal{F}$. The essential ingredients in their construction are a family of Riemann surfaces Σ, a meromorphic differential λ_{SW} on it and the definition of the prepotential in terms of period integrals of λ_{SW}

$$a_I = \int_{A_I} \lambda_{SW} \qquad \frac{\partial \mathcal{F}}{\partial a_I} = \int_{B_I} \lambda_{SW} \tag{1.1}$$

where A_I and B_I belong to a subset of the canonical cycles on the surface Σ and the a_I are a subset of the moduli parameters of the family of surfaces. These formulae define the prepotential $\mathcal{F}(a_1, ..., a_r)$ implicitly, where r denotes the rank of the gauge group under consideration.

A link between the prepotential and the Witten-Dijkgraaf-Verlinde-Verlinde equations [2],[3] was first suggested in [4]. Since then an extensive literature on the subject was formed. It was found that the perturbative piece of the prepotential $\mathcal{F}(a_1, ..., a_r)$ for pure N=2 SYM theory satisfies the generalized WDVV equations [5],[6],[7]

$$\mathcal{F}_I \mathcal{F}_K^{-1} \mathcal{F}_J = \mathcal{F}_J \mathcal{F}_K^{-1} \mathcal{F}_I \qquad \forall I, J, K = 1, ..., r \tag{1.2}$$

where the $\mathcal{F}_I$ are matrices given by $(\mathcal{F}_I)_{JK} = \frac{\partial^3 \mathcal{F}}{\partial a_I \partial a_J \partial a_K}$.

Moreover it was shown that the full prepotential for classical Lie algebras satisfies this generalized WDVV system [5],[8],[9]. The approach used by these authors consists of constructing an associative algebra of holomorphic differential forms, which together with a residue formula and existence of an invertible metric proves that the prepotential satisfies the generalized WDVV equations. For simply laced Lie algebras, an alternative proof was given in [10]. Since these include the exceptional Lie algebras of type E_6, E_7, E_8 this leaves the case of Lie algebra F_4 open[1].

In this letter, we construct the algebra of differential forms for the Lie algebra F_4 and we prove its associativity. Combined with the general remarks of [8],[9] this proves that the prepotential satisfies the WDVV equations.

[1]The algebra G_2 has rank two, which corresponds to two variables a_1, a_2. However, the generalized WDVV equations are trivial for a function $\mathcal{F}$ of only two variables.

2 Associative algebra for F_4

We start with the family of Riemann surfaces [11],[12] associated with pure F_4 Seiberg-Witten theory

$$z+\frac{\mu}{z}=W(x,u_1,...,u_4)=\frac{b_1(x)}{24}-\frac{1}{2}\left\{\left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{1/3}+\left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{1/3}\right\} \tag{2.1}$$

where p,q,b_1 are polynomials in $x,u_1,...,u_4$ which can be found in Appendix A. The Seiberg-Witten differential on this curve is

$$\lambda_{SW}=x\frac{dz}{z}=\frac{x\left(\partial_x W\right)dx}{\sqrt{W^2-4\mu}} \tag{2.2}$$

and its derivatives with respect to the moduli parameters u_i are holomorphic [9]

$$\omega_i=\frac{\partial\lambda_{SW}}{\partial u_i}\cong-\frac{\partial W}{\partial u_i}\frac{dx}{\sqrt{W^2-4\mu}}=\phi_i\frac{dx}{\sqrt{W^2-4\mu}} \tag{2.3}$$

where $\cong$ denotes equality modulo exact forms and the last equality in $(B.1)$ introduces the ϕ_i. We want to make an associative algebra out of a product structure for holomorphic differential forms

$$\omega_i\omega_j=\sum_{k=1}^{4}C_{ij}^k\omega_k G+H_{ij}\frac{dz}{z}, \tag{2.4}$$

where G is a fixed holomorphic form and H_{ij} are holomorphic forms.

There are mainly three circumstances that make the investigation different from that of the classical algebras. For Lie algebras of type A, the study of the algebra (2.4) essentially comes down to creating an algebra from a (commutative) ring of polynomials $\phi_i\in\mathbb{C}[x]$ modulo the ideal generated by the fixed polynomial $\partial_x W$. Such an algebra is automatically associative, because any commutative ring modulo an ideal is again a commutative ring. For the other classical Lie algebras, the ϕ_i and $\partial_x W$ need not be polynomial, but a multiplication by x^α for some α makes them polynomial and the same construction can be applied. In our case however (as for all the exceptional groups), this strategy does not work due to the cubic and square roots in (2.1).

Furthermore, the Riemann surfaces do not have enough known involutions, which facilitated the investigation for classical groups. These two problems will be dealt with in the following sections: we will construct a polynomial algebra in several variables and an involution of the Riemann surface is found.

Finally, the case of F_4 is more difficult from a purely computational point of view, and calculations have to be done using a computer. We did calculations in the symbolic languages REDUCE [13] and MAPLE [14].

2.1 The polynomial ring and ideals

Due to the cubic and square roots in (2.1), the $\phi_i=-\frac{\partial W}{\partial u_i}$ have terms containing $\left(-q+\sqrt{q^2+4p^3}\right)^{-\frac{2}{3}}$ which are certainly not polynomial in x. For classical gauge groups this problem does not occur and the ϕ_i are basically in a polynomial ring. It is desirable to work with a polynomial ring because it will lead to associativity of the algebra structure (2.4). For this purpose we set

$$\begin{aligned} c&=\sqrt{q^2+4p^3}\\ a&=p\left(\frac{-q+c}{2}\right)^{1/3}\\ b&=p\left(\frac{-q-c}{2}\right)^{1/3}\end{aligned} \tag{2.5}$$

and $\widetilde{\phi}_i:=abc\phi_i$. With these definitions the Riemann surface reads

$$z+\frac{\mu}{z}=\frac{b_1(x)}{24}-\frac{1}{2p}\left(a+b\right) \tag{2.6}$$

The $\widetilde{\phi}_i$ are polynomial not in one variable x, but in four variables x,a,b,c:

Proposition 1 *The $\widetilde{\phi}_i$ are elements of the polynomial ring $\mathbb{C}[x,a,b,c]$.*

Proof. See appendix B. ■

Due to the definitions of a,b,c there are certain relations among them. When multiplying the $\widetilde{\phi}_i$ to obtain an algebra, we have to take into account that

$$c^2 - q^2 - 4p^3 = 0 \tag{I.1}$$

$$ab + p^3 = 0 \tag{I.2}$$

$$a^2 - \frac{1}{2}(q-c)\, b = 0 \tag{I.3}$$

$$b^2 - \frac{1}{2}(q+c)\, a = 0 \tag{I.4}$$

These equations generate also other polynomial relations between a,b,c. For example, from the definition of a it is clear that $a^3 = \frac{1}{2}(-q+c)\, p^3$. This relation can also be deduced from $(I.3)$ and $(I.2)$:

$$a^3 = a \cdot a^2 = ab\frac{1}{2}(q-c) = \frac{1}{2}(-q+c)\, p^3 \tag{2.7}$$

We will make practical use of these relations via the following

Definition 2 *The equations (I.1),(I.2),(I.3),(I.4) generate[2] an ideal I in $\mathbb{C}[x,a,b,c]$.*

So in fact the $\widetilde{\phi}_i$ are in $\mathbb{C}[x,a,b,c]/I$ and for any equivalence class $p(x,a,b,c)+I$ of this space we can take a representative of the following form:

$$p(x,a,b,c) + I = p_1(x) + ap_2(x) + bp_3(x) + cp_4(x) + acp_5(x) + bcp_6(x) + I \tag{2.8}$$

because any higher powers of a,b,c can be rewritten using the ideal.

Since we expect the H_{ij} to be holomorphic differentials, we have taken the Ansatz that they are of the same form as the ω_i. Since

$$\omega_i = \phi_i \frac{dx}{\sqrt{W^2-4\mu}} = \frac{1}{abc}\widetilde{\phi}_i \frac{dx}{\sqrt{W^2-4\mu}} \tag{2.9}$$

with ϕ_i as in $(B.4)$, we take

$$\begin{aligned} H_{ij} &= Q_{ij}\frac{dx}{\sqrt{W^2-4\mu}} = \frac{1}{abc}\widetilde{Q}_{ij}\frac{dx}{\sqrt{W^2-4\mu}} \\ &= \frac{1}{abc}\left(abp_{ij1}(x) + ap_{ij2}(x) + bp_{ij3}(x) + abcp_{ij4}(x) + acp_{ij5}(x) + bcp_{ij6}(x)\right)\frac{dx}{\sqrt{W^2-4\mu}} \end{aligned} \tag{2.10}$$

With this choice, the classes represented by $\widetilde{Q}_{ij}$ are elements of $\mathbb{C}[x,a,b,c]/I$. Final part of the Ansatz is that the polynomials $p_{ijk}(x)$ are graded in the variables (x,u_i) with a certain degree which will be determined in section 2.2.

Furthermore we have to make a choice for the holomorphic differential G. In the ADE cases, there are reasons [10] to take $G = \omega_r = \frac{dx}{\sqrt{W^2-4\mu}}$ where r is the rank of the group. By analogy we take $G = \omega_4$.

Instead of the multiplication structure (2.4) we now look at the equivalent structure in local coordinates

$$\phi_i\phi_j = \sum_{k=1}^{4} C_{ij}^k \phi_k\phi_4 + Q_{ij}\partial_x W \tag{2.11}$$

[2] Note that defining $\widetilde{a} = \left(\frac{-q+c}{2}\right)^{1/3}$ would yield $\widetilde{a}^2 - \frac{1}{2p}(qb - bc) = 0$ which is <u>not</u> polynomial. This is the reason why an extra factor p is added in the definition of a and b.

In terms of the $\widetilde{\phi}_i$ we get

$$\left(\frac{1}{abc}\right)^2 \widetilde{\phi}_i \widetilde{\phi}_j = \left(\frac{1}{abc}\right)^2 \left(\sum_{k=1}^{4} C_{ij}^k \widetilde{\phi}_k \widetilde{\phi}_4 + \widetilde{Q}_{ij} \widetilde{\partial_x W}\right) \tag{2.12}$$

where $\widetilde{\partial_x W} = abc\partial_x W$. Equation (2.12) is equivalent to

$$\widetilde{\phi}_i \widetilde{\phi}_j = \sum_{k=1}^{4} C_{ij}^k \widetilde{\phi}_k \widetilde{\phi}_4 + \widetilde{Q}_{ij} \widetilde{\partial_x W} \tag{2.13}$$

in $\mathbb{C}[x,a,b,c]/I$. From this multiplication structure, the existence of which will be discussed in section 2.3, we can construct an algebra

Definition 3 *We define the algebra A by defining the multiplication $* : \mathbb{C}[x,a,b,c]/I \times \mathbb{C}[x,a,b,c]/I \rightarrow \mathbb{C}[x,a,b,c]/I$ by*

$$\widetilde{\phi}_i * \widetilde{\phi}_j = \sum_{k=1}^{4} C_{ij}^k \widetilde{\phi}_k \tag{2.14}$$

where the structure constants are taken from (2.13). $\widetilde{\phi}_4$ is the unity for this multiplication.

The ideal I and the ideal generated by $abc\partial_x W$ together give a new ideal J in $\mathbb{C}[x,a,b,c]$. It can be shown that $\mathbb{C}[x,a,b,c]/J$ is finite dimensional. The algebra A is obtained from polynomial multiplication modulo this ideal and it yields a 4-dimensional subalgebra of $\mathbb{C}[x,a,b,c]/J$. By our construction we have proven associativity:

Theorem 4 *The algebra A is associative.*

With this result, the problem of finding an appropriate polynomial ring has been overcome. The following section deals with a symmetry and grading of the problem.

2.2 Symmetries and grading

A very important tool in calculations is the grading which is present in the problem (see for example [11]). The origin of this grading lies in a grading of the underlying Lie algebra. We will list the degrees:

$$[x]=1 \quad , \quad [u_1]=2 \quad , \quad [u_2]=6 \quad , \quad [u_3]=8 \quad , \quad [u_4]=12$$

and from this grading we can deduce the degrees of all other objects (see appendix A). For example, $[\phi_1]=7$ so $[Q_{11}] = [\phi_1\phi_1] - [\partial_x W] = 14 - 8 = 6$ and from this we can deduce the degrees of the $p_{11k}(x)$ of equation (2.10) as promised. For H_{11} we get

$$[p_{111}]=33 \quad , \quad [p_{112}]=60 \quad , \quad [p_{113}]=60 \quad , \quad [p_{114}]=6 \quad , \quad [p_{115}]=33 \quad , \quad [p_{116}]=33$$

Apart from the involution $z \rightarrow \frac{\mu}{z}$ which is known to exist for all Riemann surfaces associated with pure Seiberg-Witten theory[11], there is at least one other involution present for F_4. As a result of the grading, we have the following involution of the Riemann surface: $z \rightarrow -z$, $x \rightarrow -x$. This involution is also exhibited by the surfaces of B_r, C_r Seiberg-Witten theory, which are constructed from the same type of procedure [11].

Under a rescaling of x by a factor α, the other objects must transform according to their degree

$$x \rightarrow \alpha x \quad , \quad z \rightarrow \alpha^9 z \quad , \quad u_1 \rightarrow \alpha^2 u_1 \quad , \quad u_2 \rightarrow \alpha^6 u_2 \quad , \quad u_3 \rightarrow \alpha^8 u_3 \quad , \quad u_4 \rightarrow \alpha^{12} u_4$$

and substituting $\alpha = -1$ gives the involution. The reason for this symmetry is therefore that x and z have odd degrees, whereas the Casimirs u_i all have even degrees. Under this symmetry, the objects transform as follows

$$\begin{array}{ll} W \mapsto -W & a \mapsto -b \\ \phi_i \mapsto -\phi_i & b \mapsto -a \\ \partial_x W \mapsto \partial_x W & c \mapsto c \end{array}$$

Using this symmetry in the multiplication structure (2.11), we find that $Q_{ij} \mapsto Q_{ij}$. This facilitates the computations by narrowing down the possible forms H_{ij}.

2.3 Construction of the algebra

The procedure we have used to calculate the multiplication structure (2.13) is to set each coefficient of the multivariate polynomial in $x, a, b, c, u_1, ..., u_4$ of the left hand side equal to that of the right hand side. This yields an overdetermined system of linear equations which can be solved uniquely to give the following structure constants[3]:

$$
\left(C_1^T\right)_j^k = \begin{pmatrix}
u_1\left(\frac{250}{243}u_1^4 - \frac{10}{9}u_1u_2 - \frac{7}{3}u_3\right) & -\frac{25}{54}u_1^3 + \frac{1}{4}u_2 & -\frac{5}{3}u_1^2 & 1 \\
\frac{100}{81}u_1^4u_2 + \frac{140}{27}u_1^3u_3 - \frac{2}{3}u_1u_2^2 - \frac{4}{3}u_1u_4 - 2u_2u_3 & u_1\left(-\frac{5}{9}u_1u_2 - \frac{7}{3}u_3\right) & -6u_3 - 2u_1u_2 & 0 \\
-\frac{2}{9}u_1u_2u_3 - \frac{2}{3}u_3^2 + \frac{100}{243}u_1^4u_3 - \frac{10}{27}u_1^2u_4 & \frac{1}{6}u_4 - \frac{5}{27}u_1^2u_3 & -\frac{2}{3}u_1u_3 & 0 \\
\frac{10}{9}u_1^2u_3^2 - \frac{1}{3}u_1u_2u_4 - u_3u_4 + \frac{50}{81}u_1^4u_4 & -\frac{1}{2}u_3^2 - \frac{5}{18}u_1^2u_4 & -u_1u_4 & 0
\end{pmatrix}
$$

$$
\left(C_2^T\right)_j^k = \begin{pmatrix}
-\frac{25}{54}u_1^3 + \frac{1}{4}u_2 & \frac{5}{24}u_1 & \frac{3}{4} & 0 \\
u_1\left(-\frac{5}{9}u_1u_2 - \frac{7}{3}u_3\right) & \frac{1}{4}u_2 & 0 & 1 \\
\frac{1}{6}u_4 - \frac{5}{27}u_1^2u_3 & \frac{1}{12}u_3 & 0 & 0 \\
-\frac{1}{2}u_3^2 - \frac{5}{18}u_1^2u_4 & \frac{1}{8}u_4 & 0 & 0
\end{pmatrix}
$$

$$
\left(C_3^T\right)_j^k = \begin{pmatrix}
-\frac{5}{3}u_1^2 & \frac{3}{4} & 0 & 0 \\
-6u_3 - 2u_1u_2 & 0 & -6u_1 & 0 \\
-\frac{2}{3}u_1u_3 & 0 & 0 & 1 \\
-u_1u_4 & 0 & -\frac{9}{2}u_3 & 0
\end{pmatrix}
$$

$$
\left(C_4^T\right)_j^k = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

Although it was proven abstractly in theorem 4 that these are structure constants of an associative algebra, this was also checked explicitly from the expressions above.

3 Conclusions and Outlook

In this letter, we constructed the algebra of holomorphic differential forms for Lie algebra F_4 and we proved its associativity. Together with the theory of [8] this proves that the prepotential of pure F_4 Seiberg-Witten theory satisfies the generalized WDVV equations. Apart from the link of Seiberg-Witten theory with integrable systems [15],[11] there is no explanation why the generalized WDVV equations should hold for this theory. One possible indication is that its origin lies in the 2D Landau-Ginzburg systems (for which the WDVV equations themselves hold), as explained for simply laced groups in [10]. In that article it was also conjectured that for B, C type Lie algebras, the generalized WDVV equations can be shown to hold by using the Landau-Ginzburg theory of BC type. This was subsequently proven in [16]. It would be interesting to use the algebra constructed explicitly in the present paper for F_4 to find out if an interpretation of the generalized WDVV equations in terms of the F_4 Landau-Ginzburg model [17] can be given.

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[3]To get a better lay-out, we give the transpose matrices $\left(C_i^T\right)_j^k = (C_i)_k^j$.

A The F_4 spectral curve

The F_4 spectral curve[4] is given by ([11],[12])

$$z + \frac{\mu}{z} = W(x, u_1, ..., u_4) = \frac{b_1(x)}{24} - \frac{1}{2}\left\{ \left(\frac{-q+\sqrt{q^2+4p^3}}{2}\right)^{1/3} + \left(\frac{-q-\sqrt{q^2+4p^3}}{2}\right)^{1/3} \right\} \tag{A.1}$$

where

$$\begin{aligned} p(x) &= -\frac{b_2}{6} - \frac{b_1^2}{144}, \\ q(x) &= \frac{1}{27}\left(\frac{b_1^3}{32} + \frac{9}{8}b_1 b_2 + 27 b_3\right) \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} b_1(x) &= -636x^9 - 300u_1x^7 - 48u_1^2x^5 - 5u_2x^3 + 2u_3x, \\ b_2(x) &= -168x^{18} - 348u_1x^{16} - 276u_1^2x^{14} + (-116u_1^3 + 14u_2)x^{12} \\ &\quad + (-92u_3 - 20u_1^4 - 8u_1u_2)x^{10} + (-42u_1u_3 - 6u_1^2u_2)x^8 \\ &\quad + (-4u_4 - \frac{10}{3}u_1^2u_3 - \frac{2}{3}u_2^2)x^6 + (\frac{1}{3}u_2u_3 - \frac{2}{3}u_4u_1)x^4, \\ b_3(x) &= x^{27} + 6u_1x^{25} + 15u_1^2x^{23} + (20u_1^3 + u_2)x^{21} + (5u_3 + 4u_1u_2 + 15u_1^4)x^{19} \\ &\quad + (6u_1^2u_2 + 12u_1u_3 + 6u_1^5)x^{17} + (\frac{1}{3}u_2^2 + 5u_4 + 4u_1^3u_2 + \frac{26}{3}u_1^2u_3 + u_1^6)x^{15} \\ &\quad + (\frac{4}{3}u_1^3u_3 + \frac{19}{3}u_4u_1 + u_1^4u_2 + \frac{4}{3}u_2u_3 + \frac{2}{3}u_2^2u_1)x^{13} \\ &\quad + (\frac{1}{3}u_1^2u_2^2 - \frac{1}{3}u_1^4u_3 - \frac{15}{4}u_3^2 + 3u_4u_1^2)x^{11} \\ &\quad + (\frac{1}{3}u_4u_2 - \frac{4}{9}u_1^2u_2u_3 + \frac{1}{27}u_2^3 - \frac{13}{6}u_3^2u_1 + \frac{13}{27}u_4u_1^3)x^9 \\ &\quad + (-\frac{1}{9}u_2^2u_3 - \frac{1}{2}u_4u_3 + \frac{1}{9}u_4u_1u_2 - \frac{7}{36}u_1^2u_3^2)x^7 + (\frac{1}{12}u_3^2u_2 - \frac{1}{6}u_4u_1u_3)x^5 \\ &\quad + (-\frac{1}{54}u_3^3 - \frac{1}{108}u_4^2)x^3. \end{aligned} \tag{A.3}$$

The degrees of several objects, induced by the grading in section 2.2, are given in the following table:

$[p]$	$[q]$	$[a]$	$[b]$	$[c]$	$[W]$	$[\partial_x W]$	$[\phi_1]$	$[\phi_2]$	$[\phi_3]$	$[\phi_4]$
18	27	27	27	27	9	8	7	3	1	-3

B Proof of proposition 1

The ϕ_i are of the form

$$\begin{aligned} \phi_i &= -\frac{\partial W}{\partial u_i} = -\frac{\partial}{\partial u_i}\left(\frac{b_1}{24} - \frac{1}{2p}(a+b)\right) = -\frac{1}{24}\frac{\partial b_1}{\partial u_i} - \frac{1}{2p^2}(a+b) + \frac{1}{2p}\frac{\partial}{\partial u_i}(a+b) \\ &= -\frac{1}{24}\frac{\partial b_1}{\partial u_i} - \frac{1}{2p^2}(a+b) + \frac{1}{2p^2}(a+b) + \\ &\qquad \frac{p^2}{6a^2}\left(-\frac{1}{2}\frac{\partial q}{\partial u_i} + \frac{1}{4c}\frac{\partial}{\partial u_i}\left(q^2+4p^3\right)\right) + \frac{p^2}{6b^2}\left(-\frac{1}{2}\frac{\partial q}{\partial u_i} - \frac{1}{4c}\frac{\partial}{\partial u_i}\left(q^2+4p^3\right)\right) \\ &= -\frac{1}{24}\frac{\partial b_1}{\partial u_i} + \frac{p^3}{6a^2}\left(-\frac{1}{2}\frac{\partial q}{\partial u_i} + \frac{q}{2c}\frac{\partial q}{\partial u_i} + \frac{3p^2}{c}\frac{\partial p}{\partial u_i}\right) + \frac{p^3}{6b^2}\left(-\frac{1}{2}\frac{\partial q}{\partial u_i} - \frac{q}{2c}\frac{\partial q}{\partial u_i} - \frac{3p^2}{c}\frac{\partial p}{\partial u_i}\right) \end{aligned} \tag{B.1}$$

[4] Note that we have corrected a misprint in [12] by adding a factor $\frac{1}{2}$ in the cube roots.

Now we use the relations $(I.1)-(I.4)$ to see that

$$
\begin{aligned}
\frac{1}{a^2} &= \frac{2}{b\,(q-c)} = \frac{2\,(q+c)}{b\,(q-c)\,(q+c)} = \frac{-\,(q+c)}{2bp^3} = -\frac{q}{2p^3b} - \frac{q^2+4p^3}{2p^3bc} \\
\frac{1}{a^2c} &= \frac{1}{c}\left(-\frac{q}{2p^3b} - \frac{q^2+4p^3}{2p^3bc}\right) = -\frac{q}{2p^3bc} - \frac{1}{2p^3b}
\end{aligned}
\tag{B.2}
$$

In a similar way one derives

$$
\begin{aligned}
\frac{1}{b^2} &= \frac{-q}{2p^3a} + \frac{q^2+4p^3}{2p^3ac} \\
\frac{1}{b^2c} &= -\frac{q}{2p^3ac} + \frac{1}{2p^3a}
\end{aligned}
\tag{B.3}
$$

Using these equations in $(B.1)$ yields

$$
\begin{aligned}
\phi_i = -\frac{1}{24}\frac{\partial b_1}{\partial u_i} &+ \frac{1}{6}\left(-\frac{q}{2b} - \frac{q^2+4p^3}{2bc}\right)\left(-\frac{1}{2}\frac{\partial q}{\partial u_i}\right) + \frac{1}{6}\left(-\frac{q}{2bc} - \frac{1}{2b}\right)\left(\frac{q}{2}\frac{\partial q}{\partial u_i} + 3p^2\frac{\partial p}{\partial u_i}\right) \\
&+ \frac{1}{6}\left(-\frac{q}{2a} + \frac{q^2+4p^3}{2ac}\right)\left(-\frac{1}{2}\frac{\partial q}{\partial u_i}\right) + \frac{1}{6}\left(-\frac{q}{2ac} + \frac{1}{2a}\right)\left(-\frac{q}{2}\frac{\partial q}{\partial u_i} - 3p^2\frac{\partial p}{\partial u_i}\right)
\end{aligned}
\tag{B.4}
$$

and therefore the $\widetilde{\phi}_i = abc\phi_i$ are polynomials in x, a, b, c.

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